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Summing Logarithms in Quantum Field Theory: The Renormalization Group

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The process of renormalization in quantum field theory necessarily involves the introduction of an arbitrary mass scale μ^2 into the theory. The effect of having this parameter appear due to quantum effects can be analyzed from many points of view; the general topic is usually called the "renormalization group." In this paper, one aspect of this feature of quantum field theory is discussed in some detail. It is shown how the appearance of this arbitrary mass scale imposes consistency conditions on quantum-induced corrections to the classical action of a model. This has the effect of determining higher order corrections in terms of lower order corrections in the perturbative expansion of the effective action, which in turn permits at least partial summation of all terms in the perturbative expansion. This is illustrated in the context of two simple, well-understood models; a ϕ^4 model in four dimensions and a ϕ^3 model in six dimensions. The technicalities associated with the renormalization procedure itself are not discussed.

1. INTRODUCTION

The process of renormalization of the parameters that characterize a theory in order to account for a physical interaction is a common occurrence. As a simple example, consider a particle suspended in the vertical direction by a spring. This system is defined in terms of three parameters; the mass m of the particle, the spring constant k, and the equilibrium position x_0. If now this system interacts with a constant external gravitational field which induces a free particle acceleration g, then the effect of this interaction is to simply "renormalize" the equilibrium position of the oscillator from x_0 to $x_0 + mg/k$ without affecting m or k.

In quantum field theory, renormalization is not only possible, but is in fact necessary in order to eliminate divergences that arise when performing

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perturbative calculations. This procedure inevitably leads to a degree of arbitrariness in any calculation, characterized by the presence of a massive parameter μ^2 in any finite perturbative result. This parameter does not have any physical significance; arbitrary changes in this scale parameter can be compensated for by finite changes in the quantities (couplings, masses, field strengths) that characterize the theory. This leads to the "renormalization group" analysis of quantum field theory (Bogoliubov and Shirkov, 1959; Callan, 1970; Gell-Mann and Low, 1954; 't Hooft, 1973; Stevenson, 1981; Stueckelberg and Peterman, 1953; Symanzik, 1970; Weinberg, 1973).

We will not discuss the technical details of how the effective action is computed or how divergent integrals are rendered meaningful through the process of "regularization." [Indeed, in an approach which uses the analytic continuation inherent in the zeta function, no explicit divergences appear in perturbation theory, either at one loop order (Salam and Strathdee, 1975; Dowker and Critchley, 1976; Hawking, 1977) or beyond (McKeon and Sherry, 1987; Culumovic *et al.*, 1989, 1990).] Our starting point will be the functional form of the effective action after the actual perturbative calculations have been done. The fundamental feature of these calculations is that when using a "mass-independent" renormalization scheme (as in 't Hooft, 1973; Weinberg, 1973; Salam and Strathdee, 1975; Dowker and Critchley, 1976; Hawking, 1977; McKeon and Sherry, 1987; Culumovic *et al.*, 1989) the coefficient of each term in the effective action is a power series in $L = \ln(m^2/\mu^2)$, where m^2 is a mass in the theory and μ^2 is the radiatively induced mass scale, which is not fixed by the form of the initial classical action.

It is often said that the renormalization group permits one to sum these logarithms. We will give an explicit demonstration of how this can be done. This gives an alternate insight into what is meant by the so-called "running coupling constants" and "running masses" that usually are defined in terms of explicitly divergent quantities that appear when deriving the "renormalization group equation."

The reason that we can perform this summation is that the coefficients of the powers of L occurring in the effective action are not all independent. The requirement that changes in μ^2 be compensated for by changes in the couplings, masses, and field strengths in the theory fixes so-called "leading logarithm" terms to all orders in perturbation theory in terms of one-loop results; "next-to-leading-order" logarithm corrections are determined by two-loop calculations, etc. (In fact, the coefficients of the leading logarithm terms indicate that they form a binomial expansion; the subleading terms are not as tractable.)

These ideas are demonstrated in the content of two simple self-interacting scalar models, a ϕ^4 model in four dimensions and a ϕ^3 model in six dimensions.

2. THE RENORMALIZATION GROUP FROM THE EFFECTIVE POTENTIAL

We will begin by discussing a ϕ^4 model in four dimensions whose classical action is

$$\Gamma_0(f) = -\frac{1}{2} f \Box f + \frac{1}{2} m^2 f^2 + \frac{1}{4!} \lambda f^4$$

where f is the classical background field. As is explained in Culumovic *et al.* (1990), the form of the effective action of this scalar model is given by

$$\begin{aligned}\Gamma(f) = &-\frac{1}{2} f \Box f [1 + \lambda(a_{11} L + a_{10}) + \lambda^2(a_{22} L^2 + a_{21} L + a_{20}) + \cdots] \\ &+ \frac{1}{2} m^2 f^2 [1 + \lambda(b_{11} L + b_{10}) + \lambda^2(b_{22} L^2 + b_{21} L + b_{20}) + \cdots] \\ &+ \frac{1}{4!} \lambda f^4 [1 + \lambda(c_{11} L + c_{10}) + \lambda^2(c_{22} L^2 + c_{21} L + c_{20}) + \cdots] \quad (1)\end{aligned}$$

where $L \equiv \ln(m^2/\mu^2)$ and the coefficients a_{mn}, b_{mn}, and c_{mn} are determined by an mth-order calculation in the perturbative loop expansion. Requiring that $\Gamma(f)$ be independent of the parameter μ^2 means that

$$\mu \frac{d\Gamma}{d\mu} = 0 \quad (2)$$

so that, by the chain rule

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} - \gamma_m(\lambda) m^2 \frac{\partial}{\partial m^2} - \gamma_\Gamma(\lambda) f \frac{\partial}{\partial f} \right] \Gamma = 0 \quad (3)$$

where

$$\beta(\lambda) = \mu \frac{\partial \lambda}{\partial \mu} \quad (4a)$$

$$-\gamma_m(\lambda) = \frac{\mu}{m^2} \frac{\partial m^2}{\partial \mu} \quad (4b)$$

and

$$-\gamma_\Gamma(\lambda) = \frac{\mu}{f} \frac{\partial f}{\partial \mu} \quad (4c)$$

This gives concrete realization to the statement in the introduction that changes in the parameters that characterize the theory compensate for the change in the radiatively induced scale parameter μ^2. If now we expand the renormalization group functions in powers of λ,

$$\beta(\lambda) = B_2\lambda^2 + B_3\lambda^3 + \cdots \tag{5a}$$

$$\gamma_m(\lambda) = G_1\lambda + G_2\lambda^2 + \cdots \tag{5b}$$

and

$$2\gamma_\Gamma(\lambda) = D_1\lambda + D_2\lambda^2 + \cdots \tag{5c}$$

then to order λ we obtain from (1), (3), and (5) that

$$D_1 = -2a_{11} \tag{6a}$$

$$G_1 = 2(a_{11} - b_{11}) \tag{6b}$$

$$B_2 = -4a_{11} + 2c_{11} \tag{6c}$$

showing that one-loop results serve to fix the lowest order contributions to β, γ_m, and γ_Γ. If we now regroup the terms in (1) so that

$$\Gamma(f) = \sum_{n=0}^{\infty} \left\{ -\frac{1}{2} f\Box f[A_n(\lambda)L^n] + \frac{1}{2} m^2 f^2[B_n(\lambda)L^n] + \frac{1}{4!} f^4[\mathbb{C}_n(\lambda)L^n] \right\} \tag{7}$$

then (3) implies that

$$(-2 - \gamma_m)nA_n + \beta A'_{n-1} - 2\gamma_\Gamma A_{n-1} = 0 \tag{8a}$$

$$(-2 - \gamma_m)nB_n + \beta B'_{n-1} - \gamma_m B_{n-1} - 2\gamma_\Gamma B_{n-1} = 0 \tag{8b}$$

$$(-2 - \gamma_m)n\mathbb{C}_n + \beta \mathbb{C}'_{n-1} + \frac{1}{\lambda}\beta \mathbb{C}_{n-1} - 4\gamma_\Gamma \mathbb{C}_{n-1} = 0 \tag{8c}$$

The expansions of A_n, B_n, and $\mathbb{C}_n$ implicit in (1), when substituted into (8), yield to order λ^n

$$a_{n,n} = \frac{1}{2n}[B_2(n - 1) - D_1]a_{n-1,n-1} \tag{9a}$$

$$b_{n,n} = \frac{1}{2n}[B_2(n - 1) - G_1 - D_1]b_{n-1,n-1} \tag{9b}$$

$$c_{n,n} = \frac{1}{2n}[B_2 n - 2D_1]c_{n-1,n-1} \tag{9c}$$

($a_{00} = b_{00} = c_{00} = 1$), which when $n = 1$ reproduce (6). For $n > 1$, (9) constitutes a set of consistency conditions that fix those contributions to $\Gamma[f]$ in (1) that are of order $\lambda^n L^n$ without explicit calculation. These are the "leading-log" contributions to $\Gamma[f]$. By iterating (9), we find that

$$a_{n,n} = \frac{1}{n!}\left[\left(\frac{B_2}{2}\right) + \left(\frac{-B_2 - D_1}{2}\right)\right]\left[\left(\frac{B_2}{2}\right)\cdot 2 + \left(\frac{-B_2 - D_1}{2}\right)\right] \times \cdots \left[\left(\frac{B_2}{2}\right) n + \left(\frac{-B_2 - D_2}{2}\right)\right] \tag{10a}$$

$$b_{n,n} = \frac{1}{n!}\left[\left(\frac{B_2}{2}\right) + \left(\frac{-B_2 - G_1 - D_1}{2}\right)\right]\left[\left(\frac{B_2}{2}\right)\cdot 2 + \left(\frac{-B_2 - G_1 - D_1}{2}\right)\right] \times \cdots \left[\left(\frac{B_2}{2}\right)\cdot n + \left(\frac{-B_2 - G_1 - D_2}{2}\right)\right] \tag{10b}$$

and

$$c_{n,n} = \frac{1}{n!}\left[\left(\frac{B_2}{2}\right) + (-D_1)\right]\left[\left(\frac{B_2}{2}\right)\cdot 2 + (-C_1)\right] \times \cdots \left[\left(\frac{B_2}{2}\right)\cdot n + (-D_1)\right] \tag{10c}$$

Since

$$(1 - \varepsilon)^{-p-1} = 1 + \frac{(p+1)}{1!}\varepsilon + \frac{(p+1)(p+2)}{2!}\varepsilon^2 + \cdots \tag{11}$$

(1) and (10) together imply that the leading-log contributions to $\Gamma[f]$ sum to

$$\Gamma_{ll}[f] = -\frac{1}{2} f\Box f\left[1 - \frac{B_2\lambda L}{2}\right]^{D_1/B_2} + \frac{1}{2} m^2 f^2\left[1 - \frac{B_2\lambda L}{2}\right]^{(D_1+G_1)/B_2} + \frac{1}{4!}\lambda f^4\left[1 - \frac{B_2\lambda L}{2}\right]^{(2D_1-B_2)/B_2} \tag{12}$$

In Culumovic *et al.* (1990), explicit calculation shows that

$$a_{11} = 0 \tag{13a}$$

$$b_{11} = \frac{1}{2}\frac{1}{(4\pi)^2} \tag{13b}$$

and

$$c_{11} = \frac{3}{2}\frac{1}{(4\pi)^2} \tag{13c}$$

so that, by (6),

$$D_1 = 0 \tag{14a}$$

$$G_1 = -\frac{1}{(4\pi)^2} \tag{14b}$$

and

$$B_2 = \frac{3}{(4\pi)^2} \tag{14c}$$

reducing (12) to

$$\begin{aligned}\Gamma_{ll}[f] = -\frac{1}{2}f\Box f + \frac{1}{2}m^2f^2\left(1 - \frac{3\lambda L}{2(4\pi)^2}\right)^{-1/3} \\ + \frac{1}{4!}\lambda f^4\left(1 - \frac{3\lambda L}{2(4\pi)^2}\right)^{-1}\end{aligned} \tag{15}$$

Hence, in the leading-log approximation, our model is characterized by an effective mass

$$m^2_{\text{eff}}(\mu^2) = \frac{m^2}{[1 - 3\lambda L/2(4\pi)^2]^{1/3}} \tag{16a}$$

and an effective coupling

$$\lambda_{\text{eff}}(\mu^2) = \frac{\lambda}{1 - 3\lambda L/2(4\pi)^2} \tag{16b}$$

These results are, in fact, solutions to the differential equations for the "running couplings" in equations (4a) and (4b) when we keep only the lowest order contributions to $\beta(\lambda)$ and $\gamma_m(\lambda)$ and impose the boundary conditions $\lambda_{\text{eff}}(m^2) = \lambda$ and $m^2_{\text{eff}}(m^2) = m^2$.

We now consider the next-to-leading-order terms in the expansion of the effective action in powers of L. By examining (8) to order λ^{n+1}, we obtain the conditions

$$\begin{aligned}-2na_{n+1,n} + (B_2n - D_1)a_{n,n-1} - G_1na_{n,n} \\ + (B_3(n-1) - D_2)a_{n-1,n-1} = 0\end{aligned} \tag{17a}$$

$$-2nb_{n+1,n} + (B_2n - G_1 - D_1)b_{n,n-1} - G_1nb_{n,n}$$

$$+ (B_3(n-1) - G_2 - D_2)b_{n-1,n-1} = 0 \tag{17b}$$

and

$$\begin{aligned} -2nc_{n+1,n} + (B_2(n+1) - 2D_1)c_{n,n-1} - G_1 n c_{n,n} \\ + (B_3 n - 2D_2)c_{n-1,n-1} = 0 \end{aligned} \tag{17c}$$

When $n = 2$ in (17), we obtain conditions that fix D_2, G_2, and B_3 in terms of the two-loop quantities a_{21}, b_{21}, c_{21}, as well the one-loop quantities a_{11}, b_{11}, c_{11} and a_{10}, b_{10}, c_{10}. With $n > 2$, consistency conditions determine $a_{n+1,n}$, $b_{n+1,n}$, $c_{n+1,n}$, but these expressions are sufficiently awkward as to prevent explicit summation of the next-to-leading-logarithms (i.e., contributions of order $\lambda^n L^{n-1}$).

The analysis of the ϕ_6^3 model can proceed along the same line. In this case the effective potential has the form (Culumovic *et al.*, 1989)

$$\begin{aligned} \Gamma[f] = \sum_{n=0}^{\infty} \Bigg\{ -\frac{1}{2} f\Box f(A_n(\lambda)L^n) + \frac{m^2}{2} f^2(B_n(\lambda)L^n) \\ + \frac{\lambda f^3}{3!}(\mathbb{C}_n(\lambda)L^n) \Bigg\} \end{aligned} \tag{18}$$

where

$$A_n(\lambda) = \sum_{m=n}^{\infty} a_{mn}\lambda^{2m} \tag{19a}$$

$$B_n(\lambda) = \sum_{m=n}^{\infty} b_{mn}\lambda^{2m} \tag{19b}$$

and

$$\mathbb{C}_n(\lambda) = \sum_{m=n}^{\infty} c_{mn}\lambda^{2m} \tag{19c}$$

in analogy with (1) and (7). The renormalization group equation

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda)\frac{\partial}{\partial \lambda} - \gamma_m(\lambda) m^2 \frac{\partial}{\partial m^2} - \gamma_\Gamma(\lambda) f \frac{\partial}{\partial f}\right]\Gamma[f] = 0 \tag{20}$$

implies that, in analogy with (8),

$$(-2 - \gamma_m) n A_n + \beta A'_{n-1} - 2\gamma_\Gamma A_{n-1} = 0 \tag{21a}$$

$$(-2 - \gamma_m) n B_n + \beta B'_{n-1} - \gamma_m B_{n-1} - 2\gamma_\Gamma B_{n-1} = 0 \tag{21b}$$

$$(-2 - \gamma_m) n B_n + \beta \mathbb{C}'_{n-1} + \frac{\beta}{\lambda} \mathbb{C}_{n-1} - 3\gamma_\Gamma \mathbb{C}_{n-1} = 0 \tag{21c}$$

If now

$$\beta(\lambda) = B_3\lambda^3 + B_5\lambda^5 + \cdots \tag{22a}$$

$$\gamma_m(\lambda) = G_2\lambda^2 + G_4\lambda^4 + \cdots \tag{22b}$$

$$2\gamma_\Gamma(\lambda) = D_2\lambda^2 + D_4\lambda^4 + \cdots \tag{22c}$$

then to order λ^{2n}, (21) gives

$$2na_{n,n} = ((2n-2)B_3 - D_2)a_{n-1,n-1} \tag{23a}$$

$$2nb_{n,n} = ((2n-2)B_3 - G_2 - D_2)b_{n-1,n-1} \tag{23b}$$

$$2nc_{n,n} = ((2n-1)B_3 - \tfrac{3}{2}D_2)c_{n-1,n-1} \tag{23c}$$

From (23) with $n = 1$ we obtain [using the results of Culumovic *et al.* (1989)]

$$B_3 = -\frac{3}{4(4\pi)^3} \tag{24a}$$

$$G_2 = \frac{5}{6}\frac{1}{(4\pi)^3} \tag{24b}$$

$$D_2 = \frac{1}{12}\frac{1}{(4\pi)^3} \tag{24c}$$

and a_{nn}, b_{nn}, c_{nn} ($n \geq 2$) are fixed by (23) in terms of B_3, G_2, D_1. In the same manner that we obtained (12), the leading-log approximation to $\Gamma[f]$ in the ϕ_6^3 model is given by

$$\begin{aligned}\Gamma_{ll}[f] = &-\tfrac{1}{2}f\Box f(1 - B_3\lambda^2 L)^{D_2/(2B_3)} \\ &+ \tfrac{1}{2}m^2 f^2(1 - B_3\lambda^2 L)^{(G_2+D_2)/(2B_3)} \\ &+ \tfrac{1}{3!}\lambda f^3(1 - B_3\lambda^2 L)^{(3D_2/4B_3)-1/2}\end{aligned} \tag{25}$$

which by (24) becomes

$$\begin{aligned}\Gamma_{ll}[f] = &-\frac{1}{2}f\Box f\left(1 + \frac{3\lambda^2 L}{4(4\pi)^3}\right)^{-1/18} \\ &+ \frac{1}{2}m^2 f^2\left(1 + \frac{3\lambda^2 L}{4(4\pi)^3}\right)^{-11/18}\end{aligned}$$

$$+ \frac{1}{3!} \lambda f^3 \left(1 + \frac{3\lambda^2 L}{4(4\pi)^3}\right)^{-7/12} \tag{26}$$

The effective mass and coupling in our ϕ_6^3 model are therefore given by

$$m_{\text{eff}}^2(\mu^2) = \frac{m^2}{[1 + 3\lambda^2 L/4(4\pi)^3]^{5/9}} \tag{27a}$$

and

$$\lambda_{\text{eff}}^2(\mu^2) = \frac{\lambda^2}{1 + 3\lambda^2 L/4(4\pi)^3} \tag{27b}$$

once we include the rescaling of the background field f implicit in the first term in (26) into the second and third terms.

As in the ϕ^4 model, the results of equation (27) can be derived by solving the differential equations (4) when only the lowest order contributions to β, γ_m, and γ_Γ are considered.

All next-to-leading logarithm contributions to the effective action are determined by the results of one- and two-loop calculations. This can be seen by examining equation (21) to order λ^{2m+2}. Again, as in the ϕ^4 model, the form of these terms does not permit one to easily sum them in closed form.

At this point we note the different sign in front of L in (16) and (27). The negative sign that appears in the former case indicates that only when L has large negative values (i.e., $\mu^2 >> m^2$) is the effective coupling sufficiently small for perturbation theory to be trustworthy. In the latter case, where the sign in front of L is positive, L must be large and positive for the effective coupling to be small (i.e., $\mu^2 << m^2$). This indicates that perturbative results in the two theories can be considered reliable in distinct momentum regimes. In the ϕ_4^4 model, momenta Q^2 must be much less than the radiatively induced scale parameter μ^2 for perturbation theory to be trusted, while for this to be true in the ϕ_6^3 model, Q^2 must exceed μ^2 by a wide margin. A simple discussion of the renormalization group from the standpoint of scale dependence is given in Stevenson (1981), and the significance of the sign in front of L in the effective parameter of the theory is reviewed there.

3. DISCUSSION

In this paper, we have considered, in the context of the two scalar models ϕ_4^4 and ϕ_6^3, the effect of demanding that changes in the renormalization scale parameter be compensated for by changes in the coupling, masses, and fields that characterize these models. We recall how this allows one to determine the renormalization group functions β, γ_m, and γ_Γ by considering the finite effective action. It is demonstrated how consistency conditions fix higher order results in terms of lower order ones. In particular, all leading-log corrections can be computed in terms of one-loop effects. This allows one to sum all leading-log contributions to the effective action in closed form. We can thereby compute an effective mass and coupling in these two models [given by equations (16) and (27)], which in fact are just the so-called running coupling and running mass functions which are solutions to equations (4a) and (4b), when we consider only lowest order contributions to the functions β, γ_m, and γ_Γ. This provides an explicit demonstration of how solving the usual renormalization group equation is equivalent to summing the logarithmic contributions to the effective action to all orders in perturbation theory.

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REFERENCES

Bogoliubov, N. N., and Shirkov, D. V. (1959). *Introduction to the Theory of Quantized Fields,* Interscience, New York.

Callan, C. G. (1970). Broken scale invariance in scalar field theory, *Physical Review D,* **3**, 1541–1547.

Culumovic, L., McKeon, D. G. C., and Sherry, T. N. (1989). Operator regularization and the renormalization group to two-loop order in ϕ_6^3, *Annals of Physics,* **197**, 94–118.

Culumovic, L., Leblanc, M., Mann, R. B., McKeon, D. G. C., and Sherry, T. N. (1990). Operator regularization and multi-loop Green's functions, *Physical Review D,* **41**, 514–533.

Dowker, J., and Critchley, R. (1976). Effective Lagrangian and energy-momentum tensor in de Sitter space, *Physical Review D,* **13**, 3224–3232.

Gell-Mann, M., and Low, F. E. (1954). Quantum electrodynamics at small distances, *Physical Review,* **95**, 1300–1312.

Hawking, S. (1977). Zeta function regularization of path integrals in curved space time, *Communications in Mathematical Physics,* **55**, 133–148.

McKeon, D. G. C., and Sherry, T. N. (1987). Operator regularization and one-loop Green's functions *Physical Review D,* **35**, 3854–3872.

Salam, A., and Strathdee, J. (1975). Transition electromagnetic fields in particle physics, *Nuclear Physics B,* **90**, 203–220.

Stevenson, P. M. (1981). Dimensional analysis in field theory, *Annals of Physics,* **132**, 383–403.

Stueckelberg, E. C. G., and Peterman, A. (1953). La normalisation des constants dans la theorie des quanta, *Helvetica Physica Acta,* **26**, 499–520.

Symanzik, K. (1970). Small distance behaviour in field theory and power counting, *Communications in Mathematical Physics,* **18**, 227–246.

't Hooft, G. (1973). Dimensional regularization and the renormalization group, *Nuclear Physics B,* **61**, 455–468.

Weinberg, S. (1973). New approach to the renormalization group, *Physical Review D,* **8**, 3497–3509.